Iwasawa L-functions for multiplicative abelian varieties. (English) Zbl 0716.14008}

Iwasawa L-functions for abelian varieties with multiplicative reductions are studied, extending some results proved by B. Mazur in Invent. Math. 18, 183-266 (1972; Zbl 0245.14015) for abelian varieties with good ordinary reductions.

Let $p \neq 2$ be a prime, $\Gamma = \mathbb{Z}_p$ (as an additive topological group) with a generator $\gamma$, and $\Lambda := \lim_{\to} \mathbb{Z}_p[\Gamma/p^nT]$. Then the map which sends $T$ to $\gamma^{-1}$ induces an isomorphism from $\mathbb{Z}_p[[T]]$ to $\Lambda$. The Iwasawa L-function for an elliptic curve was defined as the characteristic polynomial of the $p$-Selmer group of the curve. To generalize this definition to abelian varieties, one needs “good” $\Lambda$-modules which are finitely generated modules $M$ over $\Lambda$. Such a module is quasi-isomorphic to the direct sum $\Lambda^j \oplus \mathbb{Z}/p^m \mathbb{Z}[[T]] \oplus \langle r \rangle \mathbb{Z}_p[[T]](F_j)^{\nu, j}$, where $\rho$ is the free rank of $M$, $F_j$ is an irreducible distinguished polynomial for each $j$. The invariants $\langle \rho, \mu_i, \{F_j^{\nu,j}\} \rangle$ determine $M$ completely up to quasi-isomorphism (i.e., up to finite kernel and cokernel). The $\mu$-invariant of $M$ is $\mu := \sum \mu_i$, the characteristic polynomial of $M$ is $F_M(t) := p^\rho \prod_j (F_j(t))^{\nu_j}$ and $f_M(t)$ is the polynomial satisfying $f_M(T+1) = F_M(T)$.

Let $K$ be a number field with ring of integers $\mathcal{O}_K$. Let $A/K$ be an abelian variety defined over $K$, $A^0$ the connected component of $A$ and $\Gamma_0 \subset A^0$ the dual abelian variety of $A$. Assume that $A$ satisfies the following hypothesis:

1. $\text{Sh}_{p^n}(K)$ is finite.
2. Every prime of $K$ for which $A$ has bad reduction splits finitely in $L$.
3. The reduction of $A$ is semistable at every place of $K$ dividing $p$ and is an extension of an ordinary abelian variety by a torus for every $t \in T$.
4. For every place $t \in T$, the universal norm of $A(L_t)$ is of finite index in $A(K_t)$.

There is a p-adic height pairing $\langle <, >_p \rangle$ on $A$ such that $\langle <, >_p := \langle <, >_p \rangle \log_p \kappa(\gamma)$, where $\langle <, >_p$ is a p-adic height pairing defined by the author “p-adic heights for semistable abelian varieties”, Compos. Math. (to appear) and is equivalent to Schneider’s analytic height [P. Schneider, Invent. Math. 69, 401-409 (1982; Zbl 0509.14048)]. A necessary and sufficient condition for $\langle <, >_p \rangle$ to be nondegenerate is obtained. Further, define the groups $I := \text{Image}[H^1(\mathcal{O}_K, A^0_{p^n}) \rightarrow H^1(\mathcal{O}_K - T, A^0_{p^n})]$ and $I_{p^n} := \text{Image}[H^1(\mathcal{O}_L, A^0_{p^n}) \rightarrow H^1(\mathcal{O}_L - T, A^0_{p^n})]$. They are quasi-isomorphic to the classical p-Selmer group of $A$ over $K$ and $L$, respectively.

Write $A_{p^n}(L) = A^0_{p^n}(L) \oplus A^{\infty}_{p^n}(L)$ where $A^{\infty}_{p^n}$ is the divisible subgroup of $A_{p^n}(L)$. Then one can define $A^{\infty}_{p^n}(K)$ to be the K-rational points of $A^{\infty}_{p^n}(L)$. Define the $L$-invariant of $A$ with respect to $L/K$ at a place $\nu \in T$ by $L_{\nu}(A) := (A(K_\nu)/N(A(K_\nu)))/(\Phi(\mathcal{O}_K_\nu) \log_p \kappa(\gamma))^{\nu, \nu}$ and define the global $L$-invariant of $A$ with respect to $L/K$ by $L(A) := \prod_{\nu \in T} L_{\nu}(A)$.

The main result of the paper is to define a “good” $\Lambda$-module, $H$, which is subject to a quasi-exact sequence

$$0 \rightarrow I_{p^n} \rightarrow H \rightarrow (Q_p/\mathbb{Z}_p)^{c} \rightarrow 0 \quad \text{or} \quad 0 \rightarrow (Q_p/\mathbb{Z}_p)^{c} \rightarrow H \rightarrow I_{p^n} \rightarrow 0$$

where $\Gamma$ acts trivially on the $(Q_p/\mathbb{Z}_p)^{c}$ term. Let $f_H(t) = (t-1)^e f_T(t)$, and define a p-adic L-function $L_H(s) := f_H(\kappa(\gamma)^{1-s})$. (This is a candidate for the p-adic L-function of an ordinary abelian variety $A$ which is semistable at $p$.) Let $\rho = ord_{s=1} L_H(s)$ and $r = rank_{\mathbb{Z}} A(K)$. Then the main result of this paper is formulated in the following theorem:

One has $\rho \geq r + e$. If $\langle <, >_p \rangle$ is nondegenerate, then $\rho = r + e$ and the $\rho$-th derivative of $L_H(s)$ has the
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\[
L_H^0(1) \approx \mathcal{L}(A) \det <_p \frac{Sh_K}{A^m_{\infty}(K)} A^m_{\infty}(K) \prod_{\ell \in \mathcal{A}} m_{\ell},
\]

where \(m_{\ell}\) denotes the number of connected components in the fibre of \(\Lambda\) over \(\ell\) and \(a \approx b\) means that \(a\) and \(b\) have the same \(p\)-norm.

A functional equation for \(L_H(s)\) is also proved. That is, \(f_H(t) = (-1)^{p+r} f_H(1/t)\) where \(\lambda\) is the \(\lambda\)-invariant of \(\mathcal{H}\) and \(p\) is the multiplicity of the root of 1 in \(f_H(t)\), and similarly, \(L_H(s) = (-1)^{p+r} \kappa(\lambda)^{\Lambda(1-s)} L_H(2-s)\). Several candidates for such a \(\lambda\)-module are tested, e.g., \(H^1(\mathcal{O}_L, A^0_{\infty})\), \(H^1(\mathcal{O}_L, A_{\infty})\), and Greenberg’s module.

Reviewer: N. Yui

MSC:

14G10 Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture)
14K05 Algebraic theory of abelian varieties
14G40 Arithmetic varieties and schemes; Arakelov theory; heights
11G40 \(L\)-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture

Keywords:

functional equation for \(L\)-function; derivative of \(L\)-function; Birch and Swinnerton-Dyer conjecture; Iwasawa \(L\)-functions for abelian varieties with multiplicative reductions; \(p\)-adic height pairing

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References:


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