Iwasawa L-functions for multiplicative abelian varieties. (English) Zbl 0509.14048


Iwasawa L-functions for abelian varieties with multiplicative reductions are studied, extending some results proved by B. Mazur in Invent. Math. 18, 183-266 (1972; Zbl 0245.14015) for abelian varieties with good ordinary reductions.

Let \( p \neq 2 \) be a prime, \( \Gamma = \mathbb{Z}_p \) (as an additive topological group) with a generator \( \gamma \), and \( A := \varprojlim_p \mathbb{Z}_p[\Gamma/p^n\Gamma] \). Then the map which sends \( T \) to \( \gamma^{-1} \) induces an isomorphism from \( \mathbb{Z}_p[\Gamma/T] \) to \( A \). The Iwasawa L-function for an elliptic curve was defined as the characteristic polynomial of the \( p \)-Selmer group of the curve. To generalize this definition to abelian varieties, one needs “good” \( A \)-modules which are finitely generated modules \( M \) over \( A \). Such a module is quasi-isomorphic to the direct sum \( A' \oplus \mathbb{Z}/p^n\mathbb{Z}[\{T\}] \oplus (\oplus j \mathbb{Z}_p[\{T\}][F_j]^{p^n}) \) where \( p \) is the free rank of \( M \). \( F_j \) is an irreducible distinguished polynomial for each \( j \). The invariants \( \{\rho, \mu, \{F_j^{p^n}\}\} \) determine \( M \) completely up to quasi-isomorphism (i.e., up to finite kernel and cokernel). The \( \rho \)-invariant of \( M \) is \( \mu := \sum \mu_i \), the characteristic polynomial of \( M \) is \( F_M(T) := p^n \prod_j (F_j(T))^{\mu_i} \) and \( f_M(t) \) is the polynomial satisfying \( f_M(T + 1) = F_M(T) \).

Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \). Let \( A/K \) be an abelian variety defined over \( K \), \( A^0 \) the dual abelian variety of \( A \), \( \Phi \) the connected component of \( A \) and \( \tilde{A} \) be an abelian variety defined over \( K \) with \( \Phi \) its Néron model. Let \( M \) be a \( \mathbb{Z} \)-module with a free rank \( \rho \). (This is a candidate for the \( p \)-adic L-function of an ordinary abelian variety \( A \) with respect to \( K \).) Let \( \mu \) be defined by the short exact sequence \( 0 \rightarrow \mathcal{O}_K \rightarrow I \rightarrow I_p \rightarrow 0 \) where \( I_p \) is the maximal ideal of \( \mathcal{O}_K \). The \( \mu \)-invariant of \( M \) is \( \mu := \sum \mu_i \), the characteristic polynomial of \( M \) is \( F_M(T) := p^n \prod_j (F_j(T))^{\mu_i} \).

There is a p-adic height pairing \( \langle, \rangle \) and is equivalent to Schneider’s analytic height \( \langle, \rangle \), where \( \langle, \rangle \) is a p-adic height pairing defined by the author “p-adic heights for semistable abelian varieties”, Compos. Math. (to appear) and is equivalent to Schneider’s analytic height [P. Schneider, Invent. Math. 69, 401-409 (1982; Zbl 0509.14048)]. A necessary and sufficient condition for \( \langle, \rangle_p \) to be nondegenerate is obtained. Further, define the groups \( \mathcal{I} := \text{Image}[H^1(\mathcal{O}_K, A^0_p) \rightarrow H^1(\mathcal{O}_K - T, A^0_p)] \) and \( \mathcal{I}_{\infty} := \text{Image}[H^1(\mathcal{O}_K, A_{\infty}^0) \rightarrow H^1(\mathcal{O}_K - T, A_{\infty}^0)] \). They are quasi-isomorphic to the classical \( p \)-Selmer group of \( A \) over \( K \) and \( L_K \), respectively. Write \( A_{p^n}(L) = A^{\text{inf}}_{p^n}(L) \oplus A^{\text{fin}}_{p^n}(L) \) where \( A^{\text{inf}}_{p^n}(L) \) is the divisible subgroup of \( A_{p^n}(L) \). Then one can define \( A^{\text{fin}}_{p^n}(K) \) to be the \( K \)-rational points of \( A^{\text{fin}}_{p^n}(L) \). Define the \( \mathcal{L} \)-invariant of \( A \) with respect to \( L/K \) at a place \( \nu \in T \) by \( \mathcal{L}(A) := (A(K_{\nu})/NA(K_{\nu}))/(\Phi(\mathcal{O}_K_{\nu}))^{\log p} \). This global \( \mathcal{L} \)-invariant of \( A \) with respect to \( L/K \) by \( \mathcal{L}(A) := \bigoplus_{\nu \in T} \mathcal{L}(A) \).

The main result of the paper is to define a “good” \( A \)-module, \( H \), which is subject to a quasi-exact sequence

\[
0 \rightarrow \mathcal{I}_{\infty} \rightarrow H \rightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^\epsilon \rightarrow 0 \quad \text{or} \quad 0 \rightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^\epsilon \rightarrow H \rightarrow \mathcal{I}_{\infty} \rightarrow 0
\]

where \( \Gamma \) acts trivially on the \((\mathbb{Q}_p/\mathbb{Z}_p)^\epsilon\) term. Let \( f_H(t) = (t - 1)^e f_T(t) \), and define a p-adic L-function \( L_H(s) := f_H(\kappa(\gamma)^{1-e}) \). (This is a candidate for the p-adic L-function of an ordinary abelian variety \( A \) which is semistable at \( p \).) Let \( r = \text{ord}_{s=1} L_H(s) \) and \( \text{rank}_{\mathbb{Z}} A(K) \). Then the main result of this paper is formulated in the following theorem:

One has \( r \geq r + e \). If \( \langle, \rangle_p \) is nondegenerate, then \( r = r + e \) and the \( p \)-th derivative of \( L_H(s) \) has the
A functional equation for $L_H(s)$ is also proved. That is, $f_H(t) = (-1)^\rho f_H(1/t)$ where $\rho$ is the multiplicity of the root of 1 in $f_H(t)$, and similarly, $L_H(s) = (-1)^\rho \kappa(s)(1-s)L_H(2-s)$. Several candidates for such a $\lambda$-module are tested, e.g., $H^1(D_L, A_p), H^1(D_L, A_{p^\infty})$, and Greenberg’s module.

Reviewer: N. Yui

MSC:

14G10 Zeta functions and related questions in algebraic geometry (e.g., Birch-Swinnerton-Dyer conjecture)
14K05 Algebraic theory of abelian varieties
14G40 Arithmetic varieties and schemes; Arakelov theory; heights
11G40 $L$-functions of varieties over global fields; Birch-Swinnerton-Dyer conjecture

Keywords:

functional equation for L-function; derivative of L-function; Birch and Swinnerton-Dyer conjecture; Iwasawa L-functions for abelian varieties with multiplicative reductions; $p$-adic height pairing

References:


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References:


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