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Co-analytic spaces, $K$-analytic spaces, and definable versions of Menger’s conjecture. (English) [Zbl 07285211]

Topology Appl. 283, Article ID 107345, 14 p. (2020)

In the paper under review, a completely regular topological space is said to be: analytic if it is a continuous image of the space $\mathbb{P}$ of irrationals; Lusin if it is an injective continuous image of a closed subspace of $\mathbb{P}$; $K$-analytic if it is a continuous image of a Lindelöf Čech-complete space; $K$-Lusin if it is an injective continuous image of a Lindelöf Čech-complete space; co-analytic if its remainder in some compactification is analytic; co-$K$-analytic if its Stone-Čech remainder is $K$-analytic. The first result of the paper establishes that co-analytic spaces are co-$K$-analytic.

The above definitions extend, for beyond metrizable spaces, a number of notions of Descriptive Set Theory (and that we usually see at work on Polish spaces). The spirit of the paper lies in the discussion, in the realm of topological spaces, of a pattern that has been studied by analysts for more than a hundred years: “simply defined sets of reals have a nice property, e.g. Borel sets are measurable, but using the Axiom of Choice, we can construct a complicated non-measurable set”. Rather than discussing measurability, the investigation is in the field of Selection Principles: more precisely, the focus is on Menger spaces (a topological space is said to be a Menger space if whenever $\{U_n : n < \omega\}$ is a sequence of open covers, there exist finite $V_n$ such that $V_n \subseteq U_n$ for all $n < \omega$ and $\bigcup_{n<\omega} V_n$ is an open cover).

Menger’s conjecture was that Menger subsets of $\mathbb{R}$ should be $\sigma$-compact. W. Hurewicz [Math. Z. 24, 401–421 (1925; JFM 51.0454.02)] has shown that Menger analytic metrizable spaces are $\sigma$-compact, and later A. V. Arkhangel’skij [Sov. Math., Dokl. 33, 396–399 (1986; Zbl 0606.54013); translation from Dokl. Akad. Nauk SSSR 287, 525–528 (1986)] has generalized the preceding result by showing that Menger analytic spaces are $\sigma$-compact. Menger’s conjecture was disproved in ZFC by Miller and Fremlin ([Miller, A. W.; Fremlin, D. H., On some properties of Hurewicz, Menger, and Rothberger. Fundamenta Mathematicae 129, No. 1, 17-33 (1988; Zbl 0665.54026)], and they also showed that, modulo a large cardinal, it is undecidable whether Menger co-analytic sets of reals are $\sigma$-compact. In a previous work, the author and S. Tokgöz have shown [Topology Appl. 220, 111–117 (2017; Zbl 1423.54052)] that $\text{CD}$, the Axiom of Co-Analytic Determinacy – which follows from the existence of a measurable cardinal –, implies Menger co-analytic sets of reals are $\sigma$-compact – and, indeed, $\text{PD}$, the Axiom of Projective Determinacy, implies Menger projective sets of reals are $\sigma$-compact (see also [F. D. Tall, Topology Appl. 158, No. 18, 2556–2563 (2011; Zbl 1242.54009); Topology Appl. 158, No. 18, 2556–2563 (2011; Zbl 1242.54009)]). Recall that projective sets come from closed sets via a finite process of taking complements and projections.

In the paper under review, the author shows that $\text{CD}$ implies every metrizable Menger co-$K$-analytic space is $\sigma$-compact, and use this result to show that $\text{CD}$ implies every Menger co-$K$-analytical group is $\sigma$-compact. After hearing about the later result, S. Tokgöz has shown (in [Turk. J. Math. 42, No. 1, 12–20 (2018; Zbl 1424.03021)]) that under $V = L$ there is a Menger co-analytical group of reals which is not $\sigma$-compact, and thus one concludes that it is undecidable (modulo large cardinals) whether Menger co-analytical topological groups are $\sigma$-compact.

A large number of related results are established in the paper. The author also poses some questions. For instance: Are Menger $K$-Lusin spaces $\sigma$-compact? The author shows that if one strengthens Menger to Rothberger, then a positive answer is obtained (a topological space is Rothberger if whenever $\{U_n : n < \omega\}$ is a sequence of open covers, there is a cover $\{U_n : n < \omega\}$ with $U_n \subseteq U_n$ for all $n < \omega$). Are productively Lindelöf co-analytic spaces $\sigma$-compact? Productively Lindelöf spaces are those spaces $X$ whose product $X \times Y$ is Lindelöf whenever $Y$ is a Lindelöf space. Is it consistent that co-analytic Menger spaces are $\sigma$-compact? In $\text{ZFC}$, is there a co-$K$-analytic Menger space which is not $\sigma$-compact?

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MSC:
54A35 Consistency and independence results in general topology
54D45 Local compactness, \(\sigma\)-compactness
03E35 Consistency and independence results
03E75 Applications of set theory
54H05 Descriptive set theory (topological aspects of Borel, analytic, projective, etc. sets)
03E15 Descriptive set theory
03E60 Determinacy principles

Keywords:
co-analytic; Menger; \(\sigma\)-compact; productively Lindelöf; determinacy; Michael space; topological group; \(K\)-analytic; absolute Borel; \(K\)-Lusin

Full Text: DOI

References:
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[34] Tall, F. D.; Tokgöz, S., On the definability of Menger spaces which are not σ-compact, Topol. Appl., 220, 111-117 (2017) - Zbl 1425.54022

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