Efimov, Alexander I.
Homotopy finiteness of some DG categories from algebraic geometry. (English) Zbl 07286825

The author proves a conjecture of Kontsevich on homotopy finiteness of certain DG categories, which states that the bounded derived category of coherent sheaves on a separated scheme of finite type over a field \( k \) of characteristic 0 is homotopically finitely presented, by establishing a smooth categorical compactification. The basic tools in the proof are categorical resolution of singularities and the closedness of admissible subcategories of smooth proper variety under gluing via perfect bimodules. A similar result for matrix factorizations is also established in the context of \( \mathbb{Z}/2 \) graded DG categories.

The contents in more detail:

In section 1 the author gives an overview to the main content of the paper, briefly reviews some standard concepts in noncommutative algebraic geometry, states the main theorem and discuss its motivations.

In section 2 the author presents fundamental results about homotopy finiteness of DG categories and smooth categorical compactifications. Here the key concept is homotopical finite presentation (hfp) of small DG categories, which closely resembles that of finitely dominated spaces and is preserved under Drinfeld DG quotient construction.

In section 3 the author defines the notion of homological epimorphism of DG categories which generalize localization functors for small categories.

In section 4 and 5 the author discusses the gluing of DG categories via bimodules, and studies the properties of categories and functors under gluing. In particular, the gluing operation is compatible with localization and preserves hfp property.

In section 6 the author recall the notion of coderived category, which is the ind-completion of the complexes with bounded Noetherian cohomology, and the absolute derived category, which is the correct version of derived category for matrix factorization. These are backgrounds necessary for the extension of the main theorem to matrix factorization.

In section 7 the author constructs specific convenient enhancements for the category involved, and prove the versions of classical theorems for this enhancement.

The main theorem is finally proved in section 8, which roughly goes as follows: take a compactification \( Y \) of the variety we begin with, resolve its singularity by a sequence of blow up along smooth centers. By induction on the number of blow ups, the author constructs a DG category glued from smooth projective varieties with a localization functor to the derived category of \( Y \), hence establishes the main theorem. In the base step of the induction the author deals with derived categories of varieties whose reduced part is smooth by a Auslander-type construction, embedding \( D_{coh}^b(Y) \) into a compactification obtained by gluing copies of \( D_{coh}^b(Y_{red}) \), and the induction steps are proved by a categorical blow construction which partially compactify the derived category of a variety by gluing (the derived categories of) its blow up and the center. A similar result for matrix factorizations follows the same lines.

Reviewer: Kai Xu (Cambridge)

MSC:
14A30 Fundamental constructions in algebraic geometry involving higher and derived categories (homotopical algebraic geometry, derived algebraic geometry, etc.)
14E05 Rational and birational maps
18E35 Localization of categories, calculus of fractions
18G80 Derived categories, triangulated categories

Keywords:
derived categories; differential graded categories; homotopy finiteness; Verdier localization; resolution of singularities

Full Text: DOI
References:


[48] Whitehead, J. This reference list is based on information provided by the publisher or from digital mathematics libraries. Its items are heuristically matched to zbMATH identifiers and may contain data conversion errors. It attempts to reflect the references listed in the original paper as accurately as possible without claiming the completeness or perfect precision of the matching.