Let $\kappa$ and a set $X$ denote by $[X]^{<\kappa}$ the family of all subsets of $X$ of cardinality less than $\kappa$. For a topological group $(G, \tau)$ let us denote by $\tau_*$ the set of all neighbourhoods of the identity of $G$.

Inspired by classic separability, precompactness and narrowness the authors study a family of properties described by subsets of large cardinality and neighbourhoods of the identity. A sublist of these properties is the following:

Let $(G, \tau)$ be a topological group and let $\kappa, \lambda$ denote cardinals. $G$ is said to be:

- $(a)$ $u_1 s_2^1 u_1$ if for every $U_1 \in \tau_*$ there exists $S_2 \in [G]^{<\kappa}$ satisfying $U_1 S_2 U_1 = G$,
- $(b)$ $s_2^2 u_1 s_2^2$ if for every $U_1 \in \tau_*$ there exists $S_2 \in [G]^{<\kappa}$ satisfying $S_2 U_1 S_2 = G$.
- $(c)$ $u_1 s_2^1 u_3 s_3^1$ if for every $U_1 \in \tau_*$ there exists $S_2 \in [G]^{<\kappa}$ with the following property: given arbitrary $U_3 \in \tau_*$, there exists some other $S_4 \in [G]^{<\lambda}$ satisfying $(U_1 S_2)(U_3 S_4) = G$.
- $(d)$ $s_1^1 u_2 s_1^1 u_2$ if there exists $S_1 \in [G]^{<\kappa}$ such that for all $U_2 \in \tau_*$ we have $(S_1 U_2)(S_1 U_2) = G$.
- $(e)$ $u_2 s_1^1 u_2 s_3^1$ if there exists $S_1 \in [G]^{<\kappa}$ such that for all $U_2 \in \tau_*$ there exists another $S_3 \in [G]^{<\lambda}$ satisfying $(U_2 S_1)(U_2 S_3) = G$.
- $(f)$ $s_1^1 u_2 s_3^1 u_4$ if there exists $S_1 \in [G]^{<\kappa}$ such that for all $U_2 \in \tau_*$ there exists another $S_3 \in [G]^{<\lambda}$ with the following property: given arbitrary $U_4 \in \tau_*$ we have $(S_1 U_2)(S_3 U_4) = G$

In the above definition, we highlight some helpful mnemonic details:

- $(i)$ Each symbol $u_i$ denotes taking some $U_i \in \tau_*$ using a universal quantifier.
- $(ii)$ Each symbol $s_i^j$ denotes taking some $S_j \in [G]^{<\kappa}$ using an existence quantifier,
- $(iii)$ The corresponding index $i$ or $j$ in $U_i$ and $S_j^i$ indicates the precise order in which each successive step from (i) and (ii) is performed. For example: $u_1 s_2^1$ vs $u_2 s_1^1$. The former being,

$\forall U_1 \in \tau_* \quad \exists S_2 \in [G]^{<\kappa} \quad (G = U_1 S_2)$

while the latter is

$\exists S_1 \in [G]^{<\kappa} \quad \forall U_2 \in \tau_* \quad (G = U_2 S_1)$

The authors note that one may define infinitely properties by iterating the above steps. A helpful diagram of relationships between these, and many additional properties, is also provided by the authors.

Let $X$ be a set of infinite cardinality $\delta$. The main results in this paper are the following:

- The permutation group $S_{<\omega}(X)$ satisfies (a) for $\omega$, (b) for $\omega_1$ and (c) for $\kappa = \omega_1$ and $\lambda = \omega$. However, it does not satisfy neither (e) nor (f) for $\kappa = \lambda = \delta$.
- The permutation group $S(X)$ satisfies (a) for $\omega$, (d) for $\omega_1$ and (e) for $\kappa = \omega_1$ and $\lambda = \omega$. However, it does not satisfy (b) for $\kappa = \delta$.

The authors ask the following question in this paper:

(Q.) Does there exist a topological group with the $u_1 s_2^1 u_1 s_3^1 u_1$ property which does not satisfy (a) for $\omega_1$ nor (c) for $\kappa = \omega_1$ and $\lambda = \omega$?

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