For compact subsets of Banach spaces, entropies are defined as logarithms of the minimum number of balls that cover those subsets with a given radius \( \epsilon \) (there are different definitions where in one case the centres of the balls must be in the subset, and in the other case they need not; these authors use the latter). The entropy \textit{numbers} are the infimum of the radii that are allowed so that the entropy is at most a given \( k \). (In the former way of defining the entropy, those numbers about twice as big because the authors’ definition here will usually require only a smaller set of balls for a given radius, so the entropy number will be smaller than in the other definition).

Discretisations of integrals are useful to replace the continuous problem of integration by an efficient approximation through finite sums, i.e., sums of function evaluation of the integrand, perhaps by \textit{weighted} sums. These integrals may in particular come from Galerkin discretisations when partial differential equations are solved numerically or when we are computing \( L^p \)-norms of functions from finite-dimensional subspaces. In effect that means we wish to apply a cubature. It is particularly desirable to carry out this cubature by a finite, fixed set of points, and of course we are interested to know how many summands are needed. In other words, we wish to find out how many evaluations are required to put the finite sum within an \( \epsilon \)-range of the actual integral (upper and lower bounds of the sum by the exact \( L^p \)-norm, and now we take only the cases when these norms are to be estimated). This question is in the context of the present article, and the authors consider the problem from the point of view of entropies. It amounts to deriving bounds on the number of needed function evaluations (sometimes multiplied by weights) with respect to entropy numbers.

The paper puts special emphasis on so-called Marcinkiewicz discretisations and the \( \epsilon \) Marcinkiewicz discretisation. In the former, a fixed set of points is sought so that the average sum of point evaluations of the integrand at \( m \) points is \( \approx \| f \|_p \) (perhaps with weights) and in the latter the upper and lower constant factors of \( \| f \|_p \) in the estimates to the finite sums must be \( 1 \pm \epsilon \), respectively.

A special case that is particularly relevant to the authors is the case \( p = 2 \) when the aforementioned finite-dimensional subset consists of multivariable trigonometric polynomials with multi-integer exponents from a finite subset \( Q \) (so that the subspace is finite-dimensional). This may be generalised to other orthonormal bases replacing the \( \exp(ix \cdot k)s \). The case \( p = 2 \) is extended to \( p \leq 2 \), all other situations to be characteristically different.

Then, for a given \( \epsilon > 0 \), the authors prove estimates for the entropy numbers and for the required \( m \) to get an \( \epsilon \) Marcinkiewicz discretisation. They show that suitable bounds are available that use the entropy numbers. For \( N \)-dimensional subspaces, they look like \( \mathcal{O}(\epsilon N^1/3 log^2 N) \) if the entropy number is at most \( B(N/k)^{1/3} \) or the bounds can also be expressed as \( O(N \log^3 N) \) for instance. Special cases are the famous Nikolskii bounds \( \| f \|_\infty = O(N^{1/p}\| f \|_p) \) or certain bounds with respect to the Chebyshev form on the entropy numbers themselves (at most \( 62^{-k/N} \)).

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Full Text: DOI

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