Let \( R := \mathbb{K}[x_0, \ldots, x_n] \) be a graded polynomial ring over an infinite field \( \mathbb{K} \) and let \( m = \langle x_0, \ldots, x_n \rangle \).
Suppose that \( s \geq n + 1 \) and let \( \mathcal{F} = \{ f_1, \ldots, f_s \} \) be a set of forms in \( R \) satisfying some (technical) genericity conditions. For a fixed choice of positive integers \( a \) and \( b \) such that \( a \leq bs \), the generalized star configuration ideal is the uniform \( a \)-fold product ideal
\[
I_a(f^b) = I_a(f_1^b \cdots f_s^b).
\]
In general, the \( a \)-fold product ideal \( I_a(f_1^b \cdots f_s^b) \) is generated by the \( a \)-fold products of the forms \( f_1, \ldots, f_s \) with multiplicities \( b_1, \ldots, b_s \) respectively, namely
\[
I_a(f_1^{b_1} \cdots f_s^{b_s}) := \langle f_1^{n_1} \cdots f_s^{n_s} : 0 \leq n_i \leq b_i \text{ for each } i \text{ such that } \sum n_i = a \rangle.
\]
We say that an \( a \)-fold product \( I_a(f_1^{b_1} \cdots f_s^{b_s}) \) is uniform when the multiplicities \( b_1, \ldots, b_s \) are all identical, for some integer \( b \), and one writes it as \( I_a(f^b) \). It is treated as the generalized star configuration ideal since when \( b = 1 \), one will get back the star configuration ideal of hypersurfaces
\[
I_{c,F} = \bigcap_{1 \leq i_1 < \cdots < i_s \leq s} \langle f_{i_1}, \ldots, f_{i_s} \rangle
\]
for \( c = s - a + 1 \).
Recall that for a given ideal \( I \) in a ring \( R \), the \( m \)-th symbolic powers of \( I \) is defined by
\[
I^{(m)} = \bigcap_{p \in \text{Ass}(R/I)} I^m R_p \cap R.
\]
For instance, if \( I \) is the defining ideal of a reduced affine scheme over an algebraically closed field of characteristic zero, the Nagata-Zariski theorem tells us that \( I^{(m)} \) is generated by all the polynomials whose partial derivatives of orders up to \( m - 1 \) vanish on this scheme. Finally, for a homogeneous ideal \( J \) one defines \( \alpha(J) \) which is the least degree of non-zero forms in \( J \). The main result of the paper under review can be formulated as follows.

Main Theorem. Let \( a \) and \( b \) be positive integers such that \((b-1)s+1 < a \leq bs \). Assume furthermore that \( n \geq bs-a+1 \) and \( \mathcal{F} = \{ f_1, \ldots, f_s \} \) is a set of \((bs-a+1)\)-generic \( d \)-forms in \( R \), i.e., any subset of size at most \((bs-a+2)\) will form a regular sequence. Then the following properties hold for the generalized star configuration ideal \( I = I_a(f^b) \) whose big height is known to be \( h = bs-a+1 \):

i) The symbolic powers of \( I \) are sequentially Cohen-Macaulay;

ii) The ideal \( I \) satisfies the Harbourne-Huneke containment, i.e.,
\[
I^{(\ell(h+m-1)-h+k)} \subseteq m^\ell \cdot \mathbb{K}^{(\ell-1)(h-1)+k-1}(bh-(bs-a))(I^{(m)})^\ell
\]
for all positive integers \( k, \ell, m \).

iii) The inequality (being a Demailly-like bound)
\[
\frac{\alpha(I^{(\ell)})}{\ell} \geq \frac{\alpha(I^{(m)}) + h - 1}{m + h - 1}
\]
holds for all \( \ell, m \geq 1 \).

Reviewer: Piotr Pokora (Kraków)
MSC:

13A15 Ideals and multiplicative ideal theory in commutative rings
13A50 Actions of groups on commutative rings; invariant theory
13D02 Syzygies, resolutions, complexes and commutative rings
14N20 Configurations and arrangements of linear subspaces
52C35 Arrangements of points, flats, hyperplanes (aspects of discrete geometry)

Keywords:
Betti numbers; symbolic power; containment problem; star configuration; linear quotients; stable Harbourne-Huneke conjecture

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References:


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