Richter, Florian K.
A new elementary proof of the Prime Number Theorem. (English) Bull. Lond. Math. Soc. 53, No. 5, 1365-1375 (2021)

Elementary proofs of the Prime Number Theorem (PNT), does not necessarily mean simple, but refers to methods that avoid using complex analysis and instead rely only on rudimentary facts from calculus and basic arithmetic identities and inequalities. Although it was believed for a long time not to be possible, such a proof was eventually found by Erdős and Selberg, independently, both based on a Selberg’s “fundamental formula”. Later, other elementary ways of proving PNT was found by Daboussi using what he called the “convolution method” and by Hildebrand, which relies on a corollary of the large sieve.

The purpose of the paper under review is to provide yet another elementary proof of PNT, running over the comparing the mean values of the combined arithmetic functions $f(\Omega(n) + 1)$ and $f(\Omega(n))$, where $\Omega(n)$ denote the number of prime factors of a positive integer $n$ (counted with multiplicities), and $f : \mathbb{N} \to \mathbb{C}$ is any bounded arithmetic function. More precisely, the author proves that, as $N \to \infty$, one has

$$
\frac{1}{N} \sum_{n=1}^{N} f(\Omega(n) + 1) = \frac{1}{N} \sum_{n=1}^{N} f(\Omega(n)) + o(1).
$$

If we let $f(n) = (-1)^n$ then $f(\Omega(n)) = \lambda(n) = (-1)^{\Omega(n)}$ coincides with the classical Liouville function $\lambda(n)$, and from the above identity we deduce that $\sum_{n=1}^{N} \lambda(n) = o(N)$, which is a well-known equivalent form of PNT. It is worth noting that this is the first proof of PNT that builds on Chebyshev’s original idea of estimating the number of primes in the interval $(n, 2n)$.

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