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Low phase-rank approximation. (English) [Zbl 07472509]
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Summary: In this paper, we propose and solve low phase-rank approximation problems, which serve as a counterpart to the well-known low-rank approximation problem and the Schmidt-Mirsky theorem. It is well known that a nonzero complex number can be specified by its gain and phase, and while it is generally accepted that the gains of a matrix may be defined by its singular values, there is no widely accepted definition for its phases. In this work, we consider sectorial matrices, whose numerical ranges do not contain the origin, and adopt the canonical angles of such matrices as their phases. Similarly to the rank of a matrix being defined as the number of its nonzero singular values, we define the phase-rank of a sectorial matrix as the number of its nonzero phases. While a low-rank approximation problem is associated with the matrix arithmetic mean, it turns out that a natural parallel for the low phase-rank approximation problem is to use the matrix geometric mean to measure the approximation error. Importantly, we derive a majorization inequality between the phases of the geometric mean and the arithmetic mean of the phases, similarly to the Ky-Fan inequality for eigenvalues of Hermitian matrices. A characterization of the solutions to the proposed problem, with the same flavor as the Schmidt-Mirsky theorem, is then obtained in the case where both the objective matrix and the approximant are restricted to be positive-imaginary. In addition, we provide an alternative formulation of the low phase-rank approximation problem using geodesic distances between sectorial matrices. The two formulations give rise to the exact same set of solutions when the involved matrices are additionally assumed to be unitary.

MSC:
15A42 Inequalities involving eigenvalues and eigenvectors
15A45 Miscellaneous inequalities involving matrices
15A60 Norms of matrices, numerical range, applications of functional analysis to matrix theory
15B48 Positive matrices and their generalizations; cones of matrices
47A58 Linear operator approximation theory
53C22 Geodesics in global differential geometry

Keywords:
matrix phase; phase-rank; low phase-rank approximation; geometric mean; arithmetic-geometric mean inequality; majorization; geodesic distance

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References: