Summary: In this paper, we study the singular sets of $F$-subharmonic functions $u : B_2(0^n) \to \mathbb{R}$, where $F$ is a subequation. The singular set $\mathcal{S}(u) \subset B_2(0^n)$ has a stratification $\mathcal{S}^0(u) \subset \mathcal{S}^1(u) \subset \cdots \subset \mathcal{S}(u)$, where $x \in \mathcal{S}^k(u)$ if no tangent function to $u$ at $x$ is $(k+1)$-homogeneous. We define the quantitative stratifications $\mathcal{S}^k_\eta(u)$ and $\mathcal{S}^k_{\eta,r}(u)$ satisfying $\mathcal{S}^k(u) = \bigcup_{\eta} \mathcal{S}^k_\eta(u) = \bigcup_{\eta} \cap_r \mathcal{S}^k_{\eta,r}(u)$.

When homogeneity of tangents holds for $F$, we prove that $\text{dim}_H \mathcal{S}^k(u) \leq k$ and $\mathcal{S}(u) = \mathcal{S}^{n-p}(u)$, where $p$ is the Riesz characteristic of $F$. And for the top quantitative stratification $\mathcal{S}^{n-p}_\eta(u)$, we have the Minkowski estimate $\text{Vol}(B_r(\mathcal{S}^{n-p}_{\eta}(u) \cap B_1(0^n))) \leq C\eta^{-1}\left(\int_{B_1+r(0^n)} \Delta u\right)r^p$.

When uniqueness of tangents holds for $F$, we show that $\mathcal{S}^k_\eta(u)$ is $k$-rectifiable, which implies $\mathcal{S}^k(u)$ is $k$-rectifiable.

When strong uniqueness of tangents holds for $F$, we introduce the monotonicity condition and the notion of $F$-energy. By using refined covering argument, we obtain a definite upper bound on the number of $\{\Theta(u, x) \geq c\}$ for $c > 0$, where $\Theta(u, x)$ is the density of $F$-subharmonic function $u$ at $x$.

Geometrically determined subequations $F(G)$ are a very important type of subequation (when $p = 2$, homogeneity of tangents holds for $F(G)$; when $p > 2$, uniqueness of tangents holds for $F(G)$). By introducing the notion of $G$-energy and using quantitative differentiation argument, we obtain the Minkowski estimate of quantitative stratification $\text{Vol}(B_r(\mathcal{S}^k_{\eta,r}(u)) \cap B_1(0^n)) \leq C r^{n-k-p}$.

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