This article studies the direct sum of all vector bundles of conformal blocks on the moduli stack $\mathcal{M}_{g,n}$ of stable marked curves. The author demonstrates that it carries the structure of a flat sheaf of commutative algebras. The fiber over a smooth curve agrees with the Cox ring of the moduli of quasi-parabolic principal $G$-bundles on the curve for a simply connected simple complex group $G$. The heart of the paper is understanding and degenerating this object. Various techniques are applied to achieve these results and the paper relates a range of different topics such as conformal theory, representation theory of Kac-Moody Lie algebras, configuration spaces, combinatorics, tropical geometry, phylogenetics and statistical models, and integrable systems.

Let $G$ be a simply connected simple complex group and fix $B < G$ a Borel subgroup. Choose $n$ parabolic subgroups $\vec{\Lambda} = \{\Lambda_1, \ldots, \Lambda_n\}$ containing $B$. The main object of study is the moduli stack of quasi-parabolic principal $G$-bundles on a smooth complex projective curve $C$ with $\{p_1, \ldots, p_n\}$ marked points. A quasi-parabolic principal $G$-bundles of type $\vec{\Lambda}$ on $(C, \vec{p})$ is a principal $G$-bundle $E \to C$ with a choice of point $p_i$ in the fiber of $E \times_G (G/\Lambda_i)$ over $p_i$. The moduli stack of these objects is denoted by $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$ and Manon analyses in great detail its Cox ring (or total coordinate ring). Results by Faltings, Kumar-Narasimhan-Ramanathan, Beauville, Laszlo-Sorger, and Pauly identify the spaces of global sections of line bundles on $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$ with spaces of conformal blocks from the Wess-Zumino-Novikov-Witten model of conformal theory. The Cox ring $\text{Cox}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}))$ is therefore the direct sum of all such spaces of conformal blocks with compatible parabolic data.

Manon’s first main result (Theorem 1.1) treats the Cox ring of $\mathcal{M}_{C,\vec{p}}(G)$ from which the other cases can be deduced. He shows that for every trivalent graph with $n$ leaves and first Betti number equal to the genus of $C$ there exists a flat degeneration of $\text{Cox}(\mathcal{M}_{C,\vec{p}}(G))$ to certain torus invariants of a tensor product of rings $\text{Cox}(\mathcal{M}_{0,3}(G))$. This degeneration is described as a “ringification” of factorization rules for conformal blocks. The theorem opens up the perspective of obtaining a polyhedral rule for computing the famous Verlinde formula (due to Verlinde, Faltings, and Beauville) via (toric) degenerations. The Verlinde formula computes the dimensions of spaces of global sections on $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$.

When $G = \text{SL}_2(\mathbb{C})$, the flat degeneration of Theorem 1.1 is already toric as $\text{Cox}(\mathcal{M}_{0,3}(\text{SL}_2(\mathbb{C})))$ is an affine semigroup algebra. This case is treated in great detail in the last section of the paper and the source of connections to tropical geometry and phylogenetics. Manon gives further flat degenerations of $\text{Cox}(\mathcal{M}_{C,\vec{p}}(\text{SL}_2(\mathbb{C})))$ to affine semigroup algebras associated with certain weight polytopes (Theorem 1.3) and hence, toric degenerations of the coarse moduli space. These degenerations enable the author to deduce Theorem 1.5 stating that the algebra $\text{Cox}(\mathcal{M}_{C,\vec{p}}(\text{SL}_2(\mathbb{C})))$ is generated by conformal blocks of level 1 and 2 with corresponding ideal generated by relations in levels 2, 3 and 4.

Further general results regarding degenerations of the algebra of conformal blocks (Theorems 1.7 and 1.8) applied to the case of $\text{SL}_2(\mathbb{C})$ can be realized by initial degenerations in Gröbner theory associated to maximal faces of Speyer-Sturmfels tropicalization of the Grassmannian $\text{Gr}_2(\mathbb{C}^n)$. This yields Corollary 1.10 solving a conjecture by Millson regading an isomorphism between Speyer-Sturmfels degenerations associated with trivalent trees and the affine semigroup algebras associated with weight polytopes mentioned above.

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