If $R$ is a commutative ring and $A$ is an Azumaya algebra over $R$, an involution of the first kind on $A$ is an anti-automorphism of $A$ of order 2 which fixes the elements of $R$.

A. A. Albert [Structure of algebras. Providence, RI: American Mathematical Society (AMS) (1939; Zbl 0023.19901)] proved that a central simple algebra over a field admits an involution of the first kind if and only if the associated Brauer class has period at most 2. In this generality, the result does not extend to Azumaya algebras over commutative rings. However, D. J. Saltman [J. Algebra 52, 526–539 (1978; Zbl 0382.16003)] proved that a Brauer class $[A]$ over a commutative ring has period dividing 2 if and only if there is a representative $A'$ of $[A]$ that admits an involution of the first kind. Knus, Parimala, and Srinivas [M. A. Knus et al., J. Algebra 130, No. 1, 65–82 (1990; Zbl 0695.16003)] showed that in the context of Saltman’s theorem, $A'$ can be chosen so that $\deg A' = 2 \deg A$.

A question that arises naturally in the context of Saltman’s theorem is the following: Suppose $A$ has period at most 2, is $2 \deg(A)$ the least possible degree of a representative $A'$ in the Brauer class of $A$ admitting an involution of the first kind?

In the present paper, the authors construct an example that answers this question in the affirmative (Theorem A). In particular, the authors give a smooth complex algebra $R$ of finite type and an Azumaya algebra $A$ over $R$ so that (1) the degree of $A$ is 4, (2) the period of $A$ is 2, and (3) any representative $A'$ of the Brauer class of $A$ that admits an involution of the first kind has degree divisible by 8. The construction uses approximations of the universal line bundle over the classifying stack of $\text{SL}_4/\mu_2$.

Let $F$ be a field and let $A$ be a central simple algebra over $F$ of degree $n$. An involution of $F$ is called orthogonal if the dimension of $\{\sigma(a) - a : a \in A\}$ is $\frac{1}{2}n(n-1)$ and symplectic otherwise. This classification can be extended to Azumaya algebras over connected schemes [M.-A. Knus, Quadratic and Hermitian forms over rings. Berlin etc.: Springer-Verlag (1991; Zbl 0756.11008)]. For a central simple algebra of even degree over a field, it is known that the algebra either admits both types of involutions or neither. In the present paper, the authors exhibit an example that shows that this is not true for Azumaya algebras (Theorem B). They give an example of a smooth complex algebra $R$ of finite type and an Azumaya $R$-algebra $A$ so that $\deg(A) = 2$ and $A$ admits symplectic but no orthogonal involutions.

The authors review necessary background on Azumaya algebras and involutions in Section 1, and material on algebraic topology in Section 2. This section additionally includes obstructions to the existence of maps between classifying spaces, which are used in Section 3 to prove Theorem A. In Section 4 and 5, the authors show Theorem B.

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MSC:
- 14F22 Brauer groups of schemes
- 16H05 Separable algebras (e.g., quaternion algebras, Azumaya algebras, etc.)
- 16W10 Rings with involution; Lie, Jordan and other nonassociative structures
- 55R37 Maps between classifying spaces in algebraic topology
- 16K50 Brauer groups (algebraic aspects)

Keywords:
- Azumaya algebra; involution; classifying space; Brauer group; Clifford algebra; torsor; generic division algebra

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References:

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