This paper gives some asymptotic counting formulas for rational points of bounded height on anisotropic tori, in the following setting:

Let $F$ be a number field, and $T$ be a torus whose scheme of characters has no rational points, beside the trivial character. Let $X$ be toric variety with respect to $T$, and write $U \subset X$ for the dense open orbit, which is a principal homogeneous space for the torus. Furthermore, let $\mathcal{B} \subset \text{Br}(U)$ be a finite subgroup consisting of Brauer classes that vanish after base-change to the algebraic closure $F^\text{alg}$. Let $U(F)_{\mathcal{B}}$ be the ensuing set of rational points over which all members of $\mathcal{B}$ become trivial, and assume that this set in non-empty. The main result asserts that there is a constant $c > 0$ with

$$N(U, H, \mathcal{B}, B) \sim c B \frac{\rho^{-1} (\log B)}{\log B^\Delta}, \quad \text{as } B \to \infty.$$ 

The left-hand side $N(U, H, \mathcal{B}, B)$ counts the number of rational points $x \in U(F)_{\mathcal{B}}$ with bounded height $H(x) \leq B$. Here $\rho = \rho(X)$ is the Picard number of the toric variety $X$, and

$$\Delta = \sum_{D \in X^{(1)}} \left( 1 - \frac{1}{|\partial_D(\mathcal{B})|} \right)$$

is a rational number measuring the size of the finite group $\mathcal{B}$ under the residue maps at codimension-one points $D \in X$, and $H$ is the Batyrev-Tschinkel anticanonical height function [V. V. Batyrev and Y. Tschinkel, Int. Math. Res. Not. 1995, No. 12, 591–635 (1995; Zbl 0890.14008)]. It follows that the non-zero set $U(F)_{\mathcal{B}}$ is infinite, and it is actually shown to be Zariski dense.

The author also gives an interpretation of the leading constant $c = c_{X, \mathcal{B}, H}$ in terms of Artin L-functions, Tamagawa numbers, and Picard groups. This formally resembles the leading constant $c = c_{X,H,\text{Peyre}}$ conjectured to appear in the context of Manin’s Conjecture [E. Peyre, Duke Math. J. 79, No. 1, 101–218 (1995; Zbl 0901.14025)].

From the above result, a similar asymptotic formula for certain families $\pi : Y \to X$ is derived, where the restriction to $U$ becomes a product of Brauer-Severi varieties. Now one counts points on $U$, as above, but only those that lie in the image of $Y(F)$.

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