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Signature morphisms from the Cremona group over a non-closed field. (English)

The Cremona group $\text{Cr}_2(k)$ is the group of birational transformations of the projective plane $\mathbb{P}^2$ over a field $k$. It was a long-standing question whether $\text{Cr}_n(\mathbb{C})$ is simple group. Several years ago S. Cantat et al. made a breakthrough in [Acta Math. 210, No. 1, 31–94 (2013; Zbl 1278.14017)] by proving that $\text{Cr}_2(\mathbb{C})$ is not simple. A. Lonjou generalized the result to an arbitrary field in [Ann. Inst. Fourier 66, No. 5, 2021–2046 (2016; Zbl 1365.14017)]. Over the field of complex numbers it was classically known that $\text{Cr}_2(\mathbb{C})$ does not admit any non trivial homomorphism to an abelian group. Over the field of real numbers S. Zimmermann proved in [Duke Math. J. 167, No. 2, 211–267 (2018; Zbl 1402.14015)] that the abelianization of $\text{Cr}_2(\mathbb{R})$ is a direct sum of uncountably many $\mathbb{Z}/2\mathbb{Z}$.

The article under review deals with an arbitrary perfect field with at least one Galois extension of degree eight. The authors constructed a tree on which $\text{Cr}_2(k)$ acts so that $\text{Cr}_2(k)$ can be written as an amalgam product by Bass-Serre theory. Note that each factor in the amalgam product is a big group and there are a lot of factors (same cardinality as the field $k$). Consequently the authors constructed a homomorphism from $\text{Cr}_2(k)$ to a free product of $\mathbb{Z}/2\mathbb{Z}$, thus also a homomorphism from $\text{Cr}_2(k)$ to a direct sum of $\mathbb{Z}/2\mathbb{Z}$.

The tree mentioned above comes from a square complex constructed in this paper on which $\text{Cr}_2(k)$ acts. The vertices of the square are rank $r$ fibrations with $r = 1, 2, 3$; rank $r$ fibrations are generalizations of Mori fiber spaces. Roughly speaking the edges and the faces of the square complex record Sarkisov links and relations among Sarkisov links. If we blow up a general point of degree eight on $\mathbb{P}^2$ then we obtain a del Pezzo surface of degree 1. Such a del Pezzo surface gives a rank 2 fibration and an element in $\text{Cr}_2(k)$ called a Bertini involution. This is where the hypothesis on the field $k$ is used. Roughly speaking the tree is constructed by recording the action of $\text{Cr}_2(k)$ on the part of the square complex containing these Bertini involutions.

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MSC:
14E07 Birational automorphisms, Cremona group and generalizations
14E30 Minimal model program (Mori theory, extremal rays)

Keywords:
Cremona group; Sarkisov program; amalgam product

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References: