Let $S$ be an orientable surface of infinite type or a hyperbolic surface of finite type. Let $\text{Map}(S)$ be the mapping class group of $S$ and let $T(S)$ be the Teichmüller space of $S$. When $S$ of finite type, the Nielsen realization problem [J. Nielsen, Acta Math. 75, 23–115 (1942; Zbl 0027.26601)] asked whether a finite subgroup $G < \text{Map}(S)$ can be realized as a group of isometries of some hyperbolic metric on $S$. Nielsen himself showed [loc. cit.] the positivity of the result for the case when $G$ is cyclic, which was later extended by W. Fenchel [Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 5, 326–329 (1948; Zbl 0036.12902); Mat. Tidsskr. B 1950, 90–95 (1950; Zbl 0039.19303)] to the case when $G$ is solvable. However, the general case remained open for a few decades until it was finally answered in the affirmative by S. P. Kerckhoff [Ann. Math. (2) 117, 235–265 (1983; Zbl 0528.57008)]. Kerckhoff used Thurston’s earthquake deformations to show that the natural action of $G$ on $T(S)$ has a fixed point.

The study of the mapping class groups of infinite-type surfaces $S$ (also known as big mapping class groups) has gained a lot of interest [K. Ohshika (ed.) and A. Papadopoulos (ed.), In the tradition of Thurston. Geometry and topology (to appear). Cham: Springer (2020; Zbl 1470.57002), Chapter 12] in recent years. A natural question that arises in this context is whether one could derive an analog of the Nielsen realization theorem for an infinite-type surface $S$. In a recent paper [T. Aougab et al., “Isometry groups of infinite-genus hyperbolic surfaces”, Preprint, arXiv:2007.01982, to appear in Math. Ann.], it has been shown that such a result does hold true for the countable subgroups of the mapping class group for several infinite-type surfaces.

In this paper, the main result extends the Nielsen realization theorem to orientable surfaces of infinite type. A key idea in Kerckhoff’s proof lies in showing that the length function $\ell_G : T(S) \to \mathbb{R}^+$ (that sends a hyperbolic structure to the sum of the geodesic lengths of a finite $G$-invariant collection of filling curves on $S$) attains a unique minimum. But this argument does not naturally generalize to the infinite-type setting where such a length function would diverge. However, this paper adapts an alternative approach by showing the existence of an exhaustion $S_0 \subset S_1 \subset \ldots$ of connected finite-type $G$-invariant hyperbolic subsurfaces of $S$. The result is then derived by inductively applying the Nielsen realization theorem to each piece $S_k \setminus S_{k-1}$, and then carefully assembling the resultant structures in the pieces to yield a hyperbolic structure on $S$ that realizes $G$ as an isometry group.

Let $S = \mathbb{R}^2 \setminus K$, where $K$ is the Cantor set and let $S^1_C$ be the conical circle comprising geodesics originating at $\infty$. By analyzing the rotation numbers associated with the action of the torsion elements of $\text{Map}(S)$ on $S^1_C$, the following application of the main result is derived.

Theorem. Finite-order elements of $\text{Map}(S)$ fix at most one point in $K$. Moreover, for $n \geq 2$, the torsion elements in $\text{Map}(S)$ of order $n$ form at most $2\varphi(n)$ conjugacy classes, where $\varphi$ denotes the Euler totient function.

Furthermore, for an arbitrary orientable surface $S$ of infinite type, the authors show the existence of an open neighborhood of the identity in $\text{Map}(S)$ that contains no nontrivial torsion elements. This leads to another application of the main result.

Theorem. A topological group containing a sequence of nontrivial finite-order elements limiting to the identity cannot imbed in $\text{Map}(S)$.

As a final application, by proving that a compact subgroup of $\text{Map}(S)$ is virtually torsion-free consisting only of finite-order elements, they establish the following.

Theorem. The compact subgroups of $\text{Map}(S)$ are finite, and the locally compact subgroups of $\text{Map}(S)$ are discrete.

Reviewer: Kashyap Rajeevsrathy (Bhopal)