Blum, Harold; Liu, Yuchen

The normalized volume of a singularity is lower semicontinuous.  (English) Zbl 1470.14008

Let $X$ be an $n$-dimensional normal variety and let $D$ be an effective divisor on $X$. The pair $(X, D)$ is klt if $(K_X + D)$ is $\mathbb{Q}$-Cartier and, for some log resolution $f : Y \rightarrow (X, D)$, the divisor $K_Y - f^*(K_X + D)$ has all coefficients $> -1$. Given a closed point $x \in X$, we call the data $(x, X, D)$ a klt singularity. Inspired by the study of Sasaki-Einstein metrics, and motivated by the study of Kähler-Einstein metrics on Fano varieties, Chi Li introduced in [C. Li, Math. Z. 289, No. 1–2, 491–513 (2018; Zbl 1423.14025)] the notion of normalized volume $\hat{\text{vol}}(x, X, D)$ of a klt singularity $(x, X, D)$. In the paper under review, the authors show that given a $\mathbb{Q}$-Gorenstein flat family of klt singularities $(x_t, X_t, D_t)$ over a normal variety $T$, the normalized volume $\hat{\text{vol}}(x_t, X_t, D_t)$ is lower semicontinuous in $t$, with respect to the Zariski topology.

Such a result has several notable applications, either in the study of klt singularity themselves, or in the study of $\mathbb{Q}$-Fano varieties, via the affine cone construction (the affine cone over a Fano variety defined by a multiple of the anticanonical Kodaira embedding provides an archetype of klt singularity). First, a direct application shows that smooth points have the largest possible normalized volume, equal to $n^n$, among all klt singularities. This statement was also proved in [Theorem A.4, Y. Liu and Ch Xu, Duke Math. J. 168, No. 11, 2029–2073 (2019; Zbl 1436.14085)], but the paper under review proposes some generalizations along this line. Second, and maybe most importantly, the authors prove Zariski openness of log K-semistability under a conjectural hypothesis of constructibility of $t \mapsto \hat{\text{vol}}(x_t, X_t, D_t)$. The latter constructibility property was subsequently proved by Ch. Xu [Ann. Math. (2) 191, No. 3, 1003–1030 (2020; Zbl 1469.14033)]. Finally, applications to the study of Gromov-Hausdorff limits of Fano Kähler-Einstein manifolds are given, where lower semicontinuity of the normalized volume translates as lower semicontinuity of the volume density function.

The method of proof used by the authors is well described in the short introduction to the article, with a clear description of the structure of the paper.

Reviewer: Thibaut Delcroix (Montpellier)

MSC:

14B05  Singularities in algebraic geometry
13A18  Valuations and their generalizations for commutative rings
32Q20  Kähler-Einstein manifolds

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References:


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