The most fundamental of aspects in the study of topological groups is their representation theory. In a sense, this is a veritable jungle when we consider the world of all topological groups. Linear representations of finite groups inaugurated by Frobenius in the late 19th century was spectacularly continued in the next 25 years by Burnside, Schur and Weyl to cover compact groups and by Pontryagin and van Kampen to include locally compact abelian groups. Later, Gelfand and Raikov first noticed that general locally compact groups possess enough unitary representations to separate points. Later developments towards Lie groups emerged from the deep works of Bargmann, Mackey, Kirillov and Harish-Chandra.

The study of equivalence classes of unitary representations involves a diverse spectrum of notions of dual spaces. The book under review is a wonderful, masterly and exhaustive exposition. Even though the notions are discussed for general topological groups, specific attention is given to the class of discrete groups. The space of equivalence classes of its irreducible unitary representations of a topological group – called the unitary dual – is a topological space that has also the structure of a Borel space allowing us to perform direct integral decompositions of representations. Despite its crucial importance, there are situations like infinite discrete groups when this notion is far less useful as the Borel space is not countably separated.

As the authors point out in an ‘overview’, the aim of this book is to introduce and discuss the following notions of dual spaces for a group $G$:

- the unitary dual $\hat{G}$; the space $\hat{G}_{fd}$ of equivalence classes of finite dimensional irreducible representations;
- the primitive dual $Prim(G)$; the normal quasi-dual $QD(G)_{norm}$; the space of characters $Char(G)$; and
- Thoma’s dual $E(G)$.

These are all candidates for the ‘dual’ of a group and the authors exemplify pros and cons of each of these rich families through diverse examples. Through examples of the Heisenberg groups, the affine group $Aff(K)$ of a field $K$; the wreath product lamplighter group, and the general linear group over a field, they demonstrate that a full description of $\hat{G}$ for general $G$ is a hopeless task, but that a description of the spaces $\hat{G}_{fd}$, $Prim(G)$ and $E(G)$ is possible in many situations.

Now, we describe in some detail, the contents of various chapters.

In Chapter 1, the authors describe and discuss the two notions of the unitary dual $\hat{G}$ and $Prim(G)$, the primitive dual defined as the set of weak equivalence classes of irreducible unitary representations of $G$. Then, $Prim(G)$ can be thought of as the largest $T_0$-quotient of $\hat{G}$.

In Chapter 2, the representation theory of second countable, locally compact abelian groups is detailed. The main result discussed is the so-called SNAG Theorem due to Stone, Naimark, Ambrose, and Gode-ment), which shows that representations of $G$ are in bijection with projection-valued measures on $\hat{G}$.

Chapter 3 is a treasure of examples of groups and the computations of their various duals.

In Chapter 4, the authors study a comparison between $\hat{G}$ and $\hat{G}_{fd}$. In particular, questions such as when are these two spaces equal, when does the latter space separate points, when is it trivial, and when is it dense in the former are discussed in detail.

Generalizing Mackey’s theory, in Chapter 5, the authors describe for a second-countable locally compact group $G$, a general construction of all irreducible representations, up to equivalence, when $G$ is a semi-direct product of a discrete subgroup $H$ by a locally compact abelian normal closed subgroup $N$.

A Hilbert space representation $\pi_1$ of a topological group is said to be subordinate to another one $\pi_2$ if no nontrivial sub-representation of $\pi_1$ is disjoint from $\pi_2$. They are said to be quasi-equivalent if they are subordinate to each other. The quasi-dual space $QD(g)$ is then defined to be the space of quasi-equivalence classes of factor representations (representations $\pi$ for which the corresponding von Neumann algebra generated by $\pi(G)$ is a factor) of $G$. For a second countable, locally compact group $G$, the space $QD(G)_{norm}$ of quasi-equivalence classes of normal factor representations of $G$ (meaning those which are factorial and traceable) is much better behaved compared to $QD(G)$; for instance, it has a standard Borel...
structure. These are studied in depth in Chapter 10.

In this chapter, the character space $\text{Char}(G)$ of equivalence classes of characters up to multiples is also recalled. It is proved for locally compact $G$ that the characters that are defined on the maximal $C^*$-algebra of $G$ (the so-called finite characters) are in bijection with the finite part of $QD(G)$.

Earlier, in Chapter 7, the authors study the so-called groups not of type I. This includes the integral Heisenberg group, and $GL(n,K)$ for an infinite field $K$. The groups of type I are characterized by a property of their von Neumann algebras, and for discrete groups, Thoma’s characterization of type I as those which are virtually abelian, is proved in this chapter.

For the various groups studied in Chapter 3, their primitive duals are described in Chapter 9, as is the non-injectivity of the map from the unitary dual to the primitive dual for these groups.

Thoma’s dual is defined in Chapter 11, and it is computed in Chapter 12 for the discrete groups listed above.

In Chapter 11, the authors provide a nice pictorial comparison of the relations between the various duals and the maps between them.

Chapter 16 has an interesting discussion on the invariant random subgroups (IRSs) that are very topical now.

Each of the candidate notions of dual of a topological group deserves a study for its own sake. Through the careful and self-contained study of all aspects of this subject as it stands today, the authors have done just that and provided an extremely useful reference book. This is exhaustive work and will benefit researchers working on rather different aspects.

Reviewer: Balasubramanian Sury (Bangalore)

MSC:

22D10 Unitary representations of locally compact groups
22-02 Research exposition (monographs, survey articles) pertaining to topological groups
22D25 $C^*$-algebras and $W^*$-algebras in relation to group representations
22D40 Ergodic theory on groups

Keywords:

unitary dual; primitive dual; character group; topological group; LC groups; Pontryagin dual; Thoma dual; $C^*$-algebras; von Neumann algebras; representations; discrete groups; weak equivalence

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