The article under review contributes to research based around Rademacher’s theorem, which states that a Lipschitz mapping between two Euclidean spaces is differentiable almost everywhere with respect to the Lebesgue measure. More specifically, the article joins a flourishing branch of research which has the broad aim of extending Rademacher’s theorem to more general settings; see for example [J. Cheeger and B. Kleiner, Geom. Funct. Anal. 19, No. 4, 1017–1028 (2009; Zbl 1200.58007)] and [D. Bate and S. Li, Adv. Math. 333, 868–930 (2018; Zbl 1402.30047)].

Extensions of Rademacher’s theorem to infinite dimensional domains and metric measure space domains without a vector space structure must overcome various challenges. Since there is no Lebesgue like measure in these settings, such statements aim to replace the notion of a Lebesgue null set, by finding a suitable σ-ideal of null sets in the domain, in which the sets of non-differentiability points of each Lipschitz function are contained. In the case where the domain and codomain are infinite dimensional Banach spaces with some additional conditions, an appropriate σ-ideal is given by the class of Aronszajn null sets; see [N. Aronszajn, Stud. Math. 57, 147–190 (1976; Zbl 0342.46034), Theorem 1].

For functions defined on a domain without a vector space structure, the traditional notions of Fréchet and Gâteaux differentiability also need to be extended. Differentiability notions for functions defined on metric measure spaces and on Carnot groups were introduced respectively by J. Cheeger [Geom. Funct. Anal. 9, No. 3, 328–517 (1999; Zbl 0942.58018)] and P. Pansu [Ann. Math. (2) 129, No. 1, 1–60 (1989; Zbl 0678.53042)]. The article under review defines Gâteaux differentiability of a function defined on a suitable group according to the latter.

To summarise the main contribution of the paper under review, it is necessary to state a few definitions from the theory of topological groups. We define a scalable group as a pair \((G, \delta)\), where \(G\) is a topological group and \(\delta: \mathbb{R} \times G \to G\) is a continuous mapping with the following properties:

1. \(\delta_\lambda := \delta(\lambda, \cdot) \in \text{Aut}(G)\) for every \(\lambda \in \mathbb{R}\setminus \{0\}\).
2. \(\delta_\lambda \circ \delta_\mu = \delta_{\lambda \mu}\) for all \(\lambda, \mu \in \mathbb{R}\).
3. \(\delta_0 \equiv e_G\), where \(e_G\) is the identity element of \(G\).

A scalable group \((G, \delta)\) is called a metric scalable group if it admits an admissable left invariant metric \(d\), such that

\[
d(\delta_t(p), \delta_t(q)) = |t|d(p, q)
\]

for all \(t \in \mathbb{R}\) and all \(p, q \in G\). Finally a completely metric scalable group \(G\) is called an infinite dimensional Carnot group if it admits a sequence \((N_m)_{m\in\mathbb{N}}\) of scalable subgroups \(N_m\) such that each \(N_m\) has a Carnot group structure, \(N_m < N_{m+1}\) and \(G\) is the closure of \(\bigcup_{m \in \mathbb{N}} N_m\). In this case we say that \((N_m)_{m\in\mathbb{N}}\) is a filtration by Carnot subgroups of \(G\). The article under review establishes an extension of Rademacher’s theorem to infinite dimensional Carnot group domains.

The appropriate σ-ideal of null sets in an infinite dimensional Carnot group \(G\) is obtained via the notion of filtration negligible sets: Given a filtration \((N_m)_{m\in\mathbb{N}}\) by Carnot subgroups of \(G\), a Borel set \(\Omega \subseteq G\) is called \((N_m)_{m\in\mathbb{N}}\)-negligible if it is the countable union of Borel sets \(\Omega_m\) with

\[
\text{vol}_{N_m}(N_m \cap (g\Omega_m)) = 0
\]

for every \(m \in \mathbb{N}\) and every \(g \in G\). Here, \(\text{vol}_{N_m}\) denotes an arbitrarily chosen Haar measure on \(N_m\).

To conclude, the authors prove that any real-valued Lipschitz function \(f\) defined on an infinite dimensional Carnot group admits a Borel set \(\Omega \subseteq G\), such that \(f\) is Gâteaux-differentiable, in the sense of Pansu, at every point outside of \(\Omega\) and \(\Omega\) is \((N_m)_{m\in\mathbb{N}}\)-negligible with respect to every filtration \((N_m)_{m\in\mathbb{N}}\) by Carnot subgroups of \(G\).

Reviewer: Michael Dymond (Leipzig)
MSC:
28A15  Abstract differentiation theory, differentiation of set functions
46G05  Derivatives of functions in infinite-dimensional spaces
53C17  Sub-Riemannian geometry
58C20  Differentiation theory (Gateaux, Fréchet, etc.) on manifolds

Cited in 3 Documents

Keywords:
Carnot groups; differentiability; Rademacher theorem; Gâteaux derivative

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References:


[6] Wofsey, E.: (https://math.stackexchange.com/users/8686/eric wofsey), if \( h \) is a locally compact subgroup of a topological group \( G \), then \( \langle h \rangle \) is closed in \( \langle G \rangle \), Mathematics Stack Exchange. https://math.stackexchange.com/q/2419383 (version: 2017-09-06)


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